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Abstract

The increasing interest in the application of geometric stable distributions has lead to a need for appropriate estimators. Building on recent procedures for estimating the Linnik distribution, this paper develops two estimators for the geometric stable distribution $\mathrm{GS}_\alpha(\lambda, \beta, 0)$. Closed form expressions are provided for the signed and unsigned fractional moments of the distribution. The estimators are then derived using the methods of fractional lower order moments and that of logarithmic moments. Their performance is tested on simulated data, where the lower order estimators, in particular, are found to give efficient results over most of the parameter space.

Keywords: Geometric stable distribution · Estimation · Fractional lower order moments · Logarithmic moments · Economics
Mathematical Subject Classification (2000): 62-Fxx · 62-XX · 97M40

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1 Introduction

The last few years have witnessed an increasing interest in the class of geometric stable (GS) distributions (see e.g. Klebanov et al, 1985, 2006; Lekshmi and Jose, 2004). They appears as the *only* (weak) limit to a scheme of geometric sums of i.i.d. random variables, and include distributions like the Mittag-Leffler, Linnik (symmetric GS) and the Laplace distribution. Geometric sums can be expressed in the following manner,

$$S_{v_p} = Y_1 + ... + Y_{v_p}, \tag{1}$$

where the number of summed terms v_p has a geometric distribution with mean $1/p$. As the probability p goes to zero the sum tends to a GS distribution (Kozubowski and Rachev, 1999). Geometric sums and the accompanying GS laws naturally arise in a variety of applied areas, and are found to be particularly useful for modeling empirical phenomena characterized by skewness, heavy tails and marked peakedness (Kalashnikov, 1997; Kozubowski, 2001). In Kozubowski and Rachev (1994) and Mittnik and Rachev (1993) e.g., they are used to model fluctuations in financial assets; whereas in Toda and Walsh (2011) they are used to model the accumulation of consumer wealth when individuals are subjected to a death probability. Geometric compounds like (1) may also lie at the root of the ubiquitous Laplace distribution found to describe the empirical distribution of firm growth rates (Manas, 2012; Stanley et al, 1996).

Yet, the practical use of GS distributions has been limited due to the lack of a closed form expression for its density. Without an explicit density function, standard estimation procedures such as maximum likelihood estimation are highly problematic. The problem of parameter estimation has been studied in a number of previous works, in e.g. Anderson (1992), Jacques et al (1999), Kozubowski (2001) and Cahoy (2012) for the Linnik distribution and in Kozubowski (1999) for the GS distribution. Considering the Linnik distribution, Kozubowski (2001) derive estimators from solving a system of fractional lower order moment (FLOM) conditions. Using a similar approach, Cahoy (2012) solves a simple system of logarithmic moments (LM). As far as the present author is aware, Kozubowski (1999) is the only study that considers parameter estimation for the class of GS distributions. The estimators, however, are based on the empirical characteristic function and are rather difficult to implement compared to the more recent Linnik estimators.

The present paper aims to extend the symmetric Linnik estimators of Cahoy

(2012) and Kozubowski (2001) to the skewed GS distribution. Closed form expressions are derived for signed and unsigned fractional moments in terms of the parameters of the GS distribution. Two sets of estimators are then proposed, expressed as non-linear functions of the empirical FLOMs and LMs respectively. While the system of FLOMs could be solved analytically for all parameters of the centered GS distribution, the method of LM only allows for the solution of two out of three parameters. Nevertheless, the resulting estimators are easily implemented and simulation results show that the FLOM estimators, in particular, perform well over most of the parameter space.

The remainder of the paper is structured as follows. In Section 2, GS distributions are characterized and explicit expressions for its fractional moments are provided. Section 3 then gives a brief account over previous GS estimators, after which the FLOM and LM estimators are developed. Simulation results are presented in Section 4, whereas Section 5 summarizes and concludes.

2 Geometric stable distributions

2.1 Characterization

The geometric stable distribution, henceforth denoted $\mathrm{GS}_\alpha(\lambda, \beta, \mu)$, is commonly described by its characteristic function (ch.f.),

$$\phi(t) = \begin{cases} \left[1 + \lambda^\alpha |t|^\alpha \left(1 - i\beta \operatorname{sign}(t) \tan(\pi\alpha/2)\right) - i\mu t\right]^{-1}, & \text{if } \alpha \neq 1, \\ \left[1 + \lambda^\alpha |t|^\alpha \left(1 + i\beta(2\pi) \operatorname{sign}(t) \log|t|\right) - i\mu t\right]^{-1}, & \text{if } \alpha = 1, \end{cases} \tag{2}$$

where the parameter $\lambda > 0$ is a scale parameter (not to be confused with standard deviation) and where $\alpha \in (0, 2]$ is a characteristic exponent that determines the width of the distribution. For $\alpha \in (0, 2)$ the tails are slowly varying and follow a power law with cumulative distribution function $P(|X| > x) = Cx^{-\alpha}$, as $x \to \infty$. For $\alpha = 2$ the tails swiftly change as the GS distribution then collapses into the standard Laplace distribution.[1] Here, $\beta \in [-1, 1]$ is a skewness parameter. For $\beta = 0$ the distribution becomes symmetric, also known as the Linnik (or α-Laplace) distribution.[2] The parametrization in (2) is usually referred to as standard, but there exist a number of alternative parameterizations that can

[1] Thus, for $\alpha = 2$ the expression in (2) reduces to $\phi(t) = \left(1 + t^2\right)^{-1}$ which is the ch.f for the standard Laplace distribution (See Kotz et al, 2001).

[2] Setting $\beta = 0$, the resulting ch.f. becomes $\phi(t) = (1 + \lambda^\alpha |t|^\alpha)^{-1}$, which is the ch.f for the (symmetric geometric stable) Linnik distribution (c.f. Linnik, 1963; Devroye, 1990; Kotz and Ostrovskii, 1996; Pakes, 1998).

be defined via the relationship to the Lévy-stable distribution,

$$\phi(t) = (1 - \log\psi(t))^{-1}, \tag{3}$$

where $\psi(t)$ is the ch.f. of the Lévy-stable distribution (Mittnik and Rachev, 1991). Since the density of a GS distribution lacks explicit expression for $\alpha \in (0, 2)$, it needs to be estimated. Via the relationship in (3), Kozubowski (1994) shows that the distribution $\text{GS}_\alpha(\lambda, \beta, \mu)$ and density function $gs_\alpha(\lambda, \beta, \mu)$ can be calculated by solving the following integral expressions numerically,

$$\text{GS}_\alpha(x; \lambda, \beta, \mu) = \int_0^\infty \exp(-z)\,\text{LS}_\alpha\left(\frac{x - \mu z}{\lambda z^{1/\alpha}}; 1, \beta, 0\right) dz, \tag{4}$$

$$gs_\alpha(x; \lambda, \beta, \mu) = \int_0^\infty \lambda^{-1} z^{-1/\alpha} \exp(-z)\,\text{ls}_\alpha\left(\frac{x - \mu z}{\lambda z^{1/\alpha}}; 1, \beta, 0\right) dz, \tag{5}$$

where $\text{LS}_\alpha(\cdot)$ and $\text{ls}_\alpha(\cdot)$ corresponds to the distribution and density of the Lévy-stable distribution.

2.2 Fractional moments

The heavy tails of GS distributions restrict the number of existing moments q to $q < \alpha \leq 2$. In practice, this entails that only the mean ($q = 1$) usually exists out of the integer moments. For the purpose of estimation, therefore, fractional moments have been proven useful (Kozubowski, 2001; Cahoy, 2012). The unsigned and signed qth fractional moments are defined by,

$$\mu_{|x|^q} = \text{E}\left[|x|^q\right] = \int_{-\infty}^{\infty} |x|^q f(x)\, dx, \tag{6}$$

$$\mu_{x^{\langle q \rangle}} = \text{E}\left[x^{\langle q \rangle}\right] = \int_{-\infty}^{\infty} \text{sign}(x)\, |x|^q f(x)\, dx, \tag{7}$$

where $f(x)$ is the density function of some r.v. x and $\text{sign}(x)$ is the signum function. The empirical counterparts of (6) and (7) are given by,

$$\hat{\mu}_{|x|^q} = \frac{1}{n}\sum_{i=1}^{n} |x_i|^q, \tag{8}$$

$$\hat{\mu}_{x^{\langle q \rangle}} = \frac{1}{n}\sum_{i=1}^{n} \text{sign}(x_i)\, |x_i|^q. \tag{9}$$

To find $\mu_{|x|^q}$ and $\mu_{x^{\langle q \rangle}}$ when $f(x)$ is distributed after $\mathrm{GS}_\alpha(\lambda, \beta, 0)$, I follow Cahoy (2012) and use a result from Kozubowski (1994). It states that a r.v. Y with distribution function $\mathrm{GS}_\alpha(\lambda, \beta, 0)$ can be represented by a mixture of exponential and Lévy-stable distributed variables. Provided that $\alpha \neq 1$ and that Z and S are statistically independent, the following equality in distributions hold,

$$Y \stackrel{d}{=} \lambda Z^{1/\alpha} S, \tag{10}$$

where Z has an exponential distribution Exp(1) with unit scale and S a Lévy-stable distribution $\mathrm{LS}_\alpha(1, \beta, 0)$ also with unit scale, skewness parameter β and characteristic exponent α. Throughout the rest of the paper, if not mentioned otherwise, I concentrate on the centered distribution with $\mu = 0$, and $\alpha \neq 1$. The latter, when $\alpha = 1$, is considered a special case and must be treated separately.

From the structural relationship in (10) it is possible to derive closed form expressions for the signed and unsigned fractional moments of Y.

Remark 1. Let $q' = \min(1, \alpha)$, then for $q \in (-q', \alpha)$ and $\alpha \neq 1$ the qth (unsigned) fractional moment of $Y \stackrel{d}{=} \mathrm{GS}_\alpha(\lambda, \beta, 0)$ can be expressed as,

$$\mu_{|Y|^q} = \mathrm{E}\left[|Y|^q\right] = \frac{\lambda^q p\pi \cos(q\theta/\alpha)}{\alpha \sin(\pi q/\alpha)\,\Gamma(1-q)\cos(q\pi/2)\cos^{\frac{q}{\alpha}}(\theta)}, \tag{11}$$

where $\Gamma(\cdot)$ is the gamma function, and where the parameter θ is given by

$$\theta = \tan^{-1}\left(\beta \tan(\pi\alpha/2)\right). \tag{12}$$

Proof. Using (10), the qth fractional moment of Y can be written,

$$\mathrm{E}\left[|Y|^q\right] = \lambda^q \mathrm{E}\left[Z^{q/\alpha}\right] \mathrm{E}\left[|S|^q\right]. \tag{13}$$

For $\alpha \neq 1$ and $q \in (-1, \alpha)$ the qth fractional moment of the skewed Lévy-stable distribution S is provided in Kuruoglu (2001) and Nolan (1999),

$$\mathrm{E}\left[|S|^q\right] = \frac{\Gamma(1-q/\alpha)\cos(q\theta/\alpha)}{\Gamma(1-q)\cos(q\pi/2)\cos^{\frac{q}{\alpha}}(\theta)}. \tag{14}$$

The qth fractional moment of $Z^{1/\alpha}$ follows immediately from definition (6),

$$\mathrm{E}\left[Z^{q/\alpha}\right] = \int_0^\infty z^{q/\alpha} e^{-z} dz = \Gamma(1 + q/\alpha), \tag{15}$$

where $q > -\alpha$. Hence, substituting (14) and (15) into (13) results in,

$$\mathrm{E}\left[|Y|^q\right] = \lambda^q \Gamma\left(1+q/\alpha\right) \frac{\Gamma\left(1-q/\alpha\right)\cos\left(q\theta/\alpha\right)}{\Gamma\left(1-q\right)\cos\left(q\pi/2\right)\cos^{\frac{q}{\alpha}}\left(\theta\right)}, \tag{16}$$

which after using the following gamma substitutions,

$$\Gamma\left(1-q/\alpha\right) = \frac{\pi}{\Gamma\left(q/\alpha\right)\sin\left(\pi q/\alpha\right)},$$
$$\Gamma\left(1+q/\alpha\right) = \frac{q}{\alpha}\Gamma\left(q/\alpha\right),$$

results in the desired expression. □

Note that for $\beta = 0$ (and therefore also $\theta = 0$) the fractional moment equation in (11) corresponds to the expression developed in Cahoy (2012) for the symmetric GS (Linnik) distribution. By a similar argument a closed form expression for the signed fractional moment can also be found.

Remark 2. Let $q'' = \min(2, \alpha)$, then for $q \in (-q'', \alpha) \setminus \{-1\}$ and $\alpha \neq 1$ the signed qth fractional moment of $Y \sim \mathrm{GS}_\alpha(\lambda, \beta, 0)$ can be expressed as,

$$\mu_{Y^{\langle q \rangle}} = \mathrm{E}\left[Y^{\langle q \rangle}\right] = \frac{\lambda^q q\pi \sin\left(q\theta/\alpha\right)}{\alpha \sin\left(\pi q/\alpha\right)\Gamma\left(1-q\right)\sin\left(q\pi/2\right)\cos^{\frac{q}{\alpha}}\left(\theta\right)}. \tag{17}$$

Proof. Again, using the structural equation in (10) the signed qth fractional moment of Y can be written as,

$$\mathrm{E}\left[Y^{\langle q \rangle}\right] = \lambda^q \mathrm{E}\left[Z^{\langle q/\alpha \rangle}\right]\mathrm{E}\left[S^{\langle q \rangle}\right]. \tag{18}$$

For $\alpha \neq 1$ and $q \in (-2,-1) \cup (-1,\alpha)$, Kuruoglu (2001) developed the signed qth fractional moment for the skewed Lévy-stable distribution that reads,

$$\mathrm{E}\left[S^{\langle q \rangle}\right] = \frac{\Gamma\left(1-q/\alpha\right)\sin\left(q\theta/\alpha\right)}{\Gamma\left(1-q\right)\sin\left(q\pi/2\right)\cos^{\frac{q}{\alpha}}\left(\theta\right)}. \tag{19}$$

As the exponential distribution is defined only for positive real numbers the qth signed fractional moment for $Z^{1/\alpha}$ coincides with the unsigned moment (15). Hence, via (18), the results in (15) and (19) together with the cited gamma substitutions leads to the expression in (17). □

The signed and unsigned fractional moments were developed for the so called strict geometric distribution in Klebanov et al (2000), but as far as I know, the expressions developed in this section are new for the standard ch.f. in (2).

3 Estimation

The estimators developed in this section for $\mathrm{GS}_\alpha(\lambda, \beta, 0)$ are based on the expressions in (11) and (17). But for completeness, the method-of-moment (MoM) type of estimators in Kozubowski (1999), for $\mathrm{GS}_\alpha(\lambda, \beta, \mu)$, are first briefly covered. Like the Linnik estimators by Anderson (1992) and Jacques et al (1999), the MoM estimators are expressed in terms of the empirical ch.f. $\hat{\phi}_n(t) = \sum_{k=1}^{n} \exp(itY_k)$, where Y_k is a GS *i.i.d.* random variable with ch.f. given by (2). Kozubowski (1999) shows that estimates of α and λ can be calculated by solving two non-linear equations, given by

$$\mathrm{Re}\left[1/\hat{\phi}(t_k) - 1\right] = \sigma |t_k|^\alpha, \quad k = 1, 2, \tag{20}$$

where $\sigma = \lambda^\alpha$. Here, the notation $\mathrm{Re}[\cdot]$ refers to the real part of the expression. Similarly, estimates of β and μ can be calculated by solving the following system,

$$-\mathrm{Im}\left[1/\hat{\phi}(t_k)\right] = \mu t_k + \sigma |t_k|^\alpha \beta \tan(\pi\alpha/2)\,\mathrm{sign}(t_k), \quad k = 1, 2, \tag{21}$$

given previous estimates of α and λ, where $\mathrm{Im}[\cdot]$ stands for the imaginary part of the reciprocal empirical ch.f. The estimators are found to be consistent. However, it is not clear how the appropriate sequence of $\{t_k\}$ should be chosen, which is likely to put restrictions on their practical use.[3]

3.1 The method of fractional lower order moments

The method of finding estimators by expressing the parameters of a distribution in terms of its fractional lower order moments was previously used by e.g. Kozubowski (2001) for the Linnik distribution, and Kuruoglu (2001) for the skewed Lévy-stable distribution. Here, I adopt the technique in Kuruoglu (2001) where parameters are expressed in terms of non-linear functions of $\mu_{\pm q}$ and $\mu_{\langle \pm q \rangle}$. Estimates are then found by substituting for the empirical analogs $\hat{\mu}_{\pm q}$ and $\hat{\mu}_{\langle \pm q \rangle}$, defined in (8) and (9). For some r.v. $Y \overset{d}{=} \mathrm{GS}_\alpha(\lambda, \beta, 0)$ it turns

[3] Kozubowski (1999) also introduce estimators based on ordinary least squares. But, since this method is similar to MoM it is not covered here.

out that α can be estimated by inverting the following sinc function,

$$\begin{aligned}\operatorname{sinc}\left(\frac{q\pi}{\alpha_{FLOM}}\right) &= \left(\left(\hat{\mu}_{|Y|^q}\hat{\mu}_{|Y|^{-q}}\cot\left(\frac{q\pi}{2}\right)\right.\right. \\ &\quad \left.\left.+\hat{\mu}_{Y\langle q\rangle}\hat{\mu}_{Y\langle -q\rangle}\left(\pi\alpha/2\right)\tan\left(\frac{q\pi}{2}\right)\right)\frac{q\pi}{2}\right)^{-\frac{1}{2}}. \qquad (22)\end{aligned}$$

A caveat with the FLOM method is that efficiency depends on the choice of q, which is confined to the parameter space explicated in the previous section. The appropriate q for estimating α is examined in Section 4, and found to be $q = 0.2$ (c.f. Ma and Nikias, 1995).[4]

For the skewness parameter β it is possible to use the ratio of $\mu_{Y\langle q\rangle}$ and $\mu_{|Y|^q}$ to arrive at an estimator. Given estimates of θ and α, β can be estimated via

$$\beta_{FLOM} = \frac{\tan\left(\theta\right)}{\tan\left(\frac{\alpha\pi}{2}\right)}, \qquad (23)$$

which follows from (12), and where θ is estimated by

$$\theta_{FLOM} = \frac{\alpha}{q}\tan^{-1}\left(\left(\hat{\mu}_{Y\langle q\rangle}/\hat{\mu}_{|Y|^q}\right)\tan\left(\frac{q\pi}{\alpha}\right)\right). \qquad (24)$$

Once estimates of α, β and θ have been obtained, λ can then be estimated directly from (11), by solving for,

$$\lambda_{FLOM} = \left(\frac{\alpha\sin\left(\pi q/\alpha\right)\Gamma\left(1-q\right)\cos\left(q\pi/2\right)\cos^{\frac{q}{\alpha}}\left(\theta\right)}{q\pi\cos\left(q\theta/\alpha\right)}\hat{\mu}_{|Y|^q}\right)^{\frac{1}{q}}. \qquad (25)$$

The appropriate q for $\hat{\beta}_{FLOM}$ and $\hat{\lambda}_{FLOM}$, is located at $q \to 0$ (See Section 4). This method for choosing q differs from Kozubowski (2001) who propose the choice of $q_1 = 1/2$ and $q_2 = 1$ for the Linnik distribution $\mathrm{GS}_\alpha\left(\lambda, 0, 0\right)$. In the context of skewed GS distributions no support is found for these particular choices of q_i.

3.2 The method of logarithmic moments

In contrast to the FLOM method, logarithmic moments (LM) ensures the existence of higher order moments. For GS distributions, parameter estimates can

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[4]Note that for $\beta = 0$ the signed fractional moments $\mu_{Y\langle q\rangle}$ in (17) turns zero. This result is fully compatible with the sinc estimator in (22), which for $\theta = 0$ reduces to $\operatorname{sinc}\left(q\pi/\alpha_{FLOM}\right) = \left(\hat{\mu}_{|Y|^q}\hat{\mu}_{|Y|^{-q}}\cot\left(q\pi/2\right)q\pi/2\right)^{-1/2}$. This formula is nothing less than the FLOM estimator of α for the Linnik distribution

therefore, be expressed as functions of integer moments $q = 1, ..., n$. Thus, no prior choice of q is therefore needed. This method has previously been used to estimate the parameters of e.g. the skewed Lévy-stable distribution (Kuruoglu, 2001), the fractional stable distribution (Bening et al, 2004) and the fractional Poisson process (Cahoy et al, 2010). To find LM estimators for $\mathrm{GS}_\alpha(\lambda, \beta, 0)$, I attempt to extend the approach in Cahoy (2012) who developed estimators for the Linnik distribution based on a simple system of LM conditions.

In contrast to the symmetric Linnik distribution, however, no closed form solution is found for the complete system of LM conditions. Given knowledge of α, estimates for the skewness and scale parameters β and λ are nonetheless provided.

To find the logarithmic (unsigned) moments it suffice to calculate the derivative of (11), using the result,

$$\mathrm{E}\left[\log |Y|^n\right] = \lim_{q \to 0} \frac{\mathrm{d}^n}{\mathrm{d}q^n} \mathrm{E}\left[|Y|^q\right]. \tag{26}$$

But, as shown in Bening et al (2004) and Cahoy (2012), finding $\mathrm{E}\left[\log |Y|^n\right]$ turns out to be easier if $\mu_{|Y|^q}$ is first expand into a power series. Taking the logarithm of (11) results in,

$$\begin{aligned} \log \mu_{|Y|^q} = \; & q \log \lambda + \log q\pi + \log \cos\left(q\theta/\alpha\right) \\ & - \log \alpha - \log \sin\left(\pi q/\alpha\right) - \log \Gamma\left(1-q\right) \\ & - \log \cos\left(q\pi/2\right) - \log \cos^{\frac{q}{\alpha}}\left(\theta\right). \end{aligned} \tag{27}$$

Then using the following series expansions around $q = 0$ up to order $\mathrm{O}\left(q^4\right)$,

$$\log \cos\left(q\theta/\alpha\right) = \frac{\theta^2}{2\alpha^2} q^2 + \mathrm{O}\left(q^4\right),$$

$$\log \sin\left(\pi q/\alpha\right) = \log q + \log \frac{\pi}{\alpha} - \frac{\pi^2 q^2}{6\alpha^2} + \mathrm{O}\left(q^4\right),$$

$$\log \Gamma\left(1-q\right) = \mathbb{C} q + \frac{\pi^2 q^2}{12} + \frac{1}{3}\zeta_{(3)} q^3 + \mathrm{O}\left(q^4\right),$$

gives the expression,

$$\begin{aligned}\log\mu_{|Y|^q} &= \left(\log(\lambda) - \mathbb{C} - \frac{1}{\alpha}\log\cos(\theta)\right) q \\ &+ \frac{(4+\alpha^2)\pi^2 - 12\theta^2}{24\alpha^2} q^2 \\ &- \frac{1}{3}\zeta_{(3)} q^3 + \mathrm{O}(q^4),\end{aligned} \tag{28}$$

where $\mathbb{C}$ is the is the Euler–Mascheroni constant

$$\mathbb{C} = \lim_{n\to\infty}\left(\sum_{j=1}^{n}\frac{i}{1} - \log n\right) = \Gamma(-1) = 0.577215664901... \tag{29}$$

Taking the exponential function of (28), and once again expanding around $q = 0$, finally results in the following power series,

$$\begin{aligned}\mu_{|Y|^q} &= 1 + Kq + \left[\frac{(4+\alpha^2)\pi^2 - 12\theta^2}{24\alpha^2} + \frac{1}{2}K\right] q^2 \\ &+ \left[\frac{(4+\alpha^2)\pi^2 - 12\theta^2}{24\alpha^2} K + \frac{1}{6}K^3 - \frac{1}{3}\zeta_{(3)}\right] q^3 \\ &+ \mathrm{O}\left(q^4\right),\end{aligned} \tag{30}$$

where $K = \log(\lambda) - \mathbb{C} - \frac{1}{\alpha}\log\cos(\theta)$ to simplify notations. Thus, finding the nth LM is now straight forward, and amounts to evaluating the nth derivative of (30) at the limit, when $q \to 0$.

The first three central moments of $\log|Y|$ can then be readily calculated,

$$m_1 = \mathrm{E}\left[\log|Y|^1\right] = \log(\lambda) - \mathbb{C} - \frac{1}{\alpha}\log\cos(\theta), \tag{31}$$

$$m_2 = \mathrm{E}\left[(\log|Y| - \mathrm{E}[\log|Y|])^2\right] = \frac{(4+\alpha^2)\pi^2 - 12\theta^2}{12\alpha^2}, \tag{32}$$

$$m_3 = \mathrm{E}\left[(\log|Y| - \mathrm{E}[\log|Y|])^3\right] = -2\zeta_{(3)}, \tag{33}$$

where $\zeta_{(3)}$ is the Zeta function evaluated at 3. For the symmetric situation when $\beta = 0$ the system (31-33) coincides with the moment conditions reached in Cahoy (2012), which can be solved analytically for α and λ in term of the first and second central moment of $\log|Y|$,

$$\alpha = \frac{2\pi}{\sqrt{12\hat{m}_2 - \pi^2}}, \tag{34}$$

$$\lambda = \exp\left(\hat{m}_1 + \mathbb{C}\right). \tag{35}$$

Cahoy (2012) also shows that the resulting estimates are both consistent and asymptotically normal.

For the skewed distribution, when $\beta \neq 0$, the situation becomes more complicated, as the system lacks analytical solution. Calculating still higher moments are possible but does not offer a solution, and even if found higher moments are usually found to be shaky (Cahoy, 2012).

One possible remedy would be to first estimate α by the FLOM method and then solve for the remaining β and λ. Hence, given an estimate of α the system in (31-32) can be solved for

$$\beta_{LM} = \pm\cot\left(\frac{\pi\alpha}{2}\right)\tan\left(\frac{1}{2}\sqrt{\frac{1}{3}\pi^2\left(\alpha^2+4\right) - 4\alpha^2\hat{m}_2}\right), \tag{36}$$

and

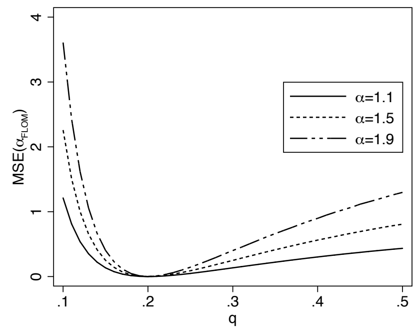
$$\lambda_{LM} = \exp\left(\hat{m}_1 + \mathbb{C}\right)\cos\left(\theta\right)^{1/\alpha}, \tag{37}$$

where $|\theta|$ is estimated from $|\theta| = \tan^{-1}\left(\beta\tan\left(\frac{\pi\alpha}{2}\right)\right)$. In the resulting estimators, $\hat{m}_1$ and $\hat{m}_2$ corresponds to the estimated mean and variance of $\log|Y|$. A shortcoming with the LM method, however, is that it is not possible to tell the sign of the β parameter. Instead, one must resort to examining the underlying data, or look at the FLOM estimate of β.

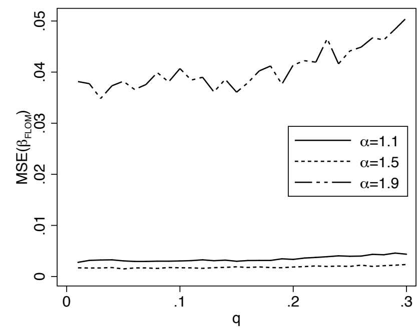
4 Performance of the estimators on simulated data

In this section the estimators are tested on simulated data. All simulations are based on 1000 samples with each n=10000 observations drawn from a GS distribution. The random variables are created from the representation in (10), with $\lambda = 1$ and $\mu = 0$ for various α and β.

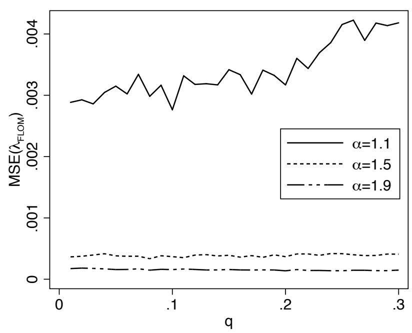
In the FLOM method the appropriate order of the fractional moments q must first be determined. The results for q-dependence are displayed in Figure 1 that shows the mean squared error (MSE) as a function of q. In Figure 1(a), MSE for $\hat{\alpha}_{FLOM}$ is minimized at $q = 0.2$. Here, β is set to 0.9, but the result is robust



(a) MSE of $\hat{\alpha}_{FLOM}$ for different q



(b) MSE of $\hat{\beta}_{FLOM}$ for different q



(c) MSE of $\hat{\lambda}_{FLOM}$ for different q

Figure 1. Simulation results over q-dependence with 1000 simulations with $n = 10000$, $\beta = 0.9$ and $\lambda = 1$

throughout the parameter space of β. This regularity steam from the Lévy-stable distribution, for which Ma and Nikias (1995) finds the same appropriate choice of q. For the estimators $\hat{\beta}_{FLOM}$ and $\hat{\lambda}_{FLOM}$, on the other hand, MSE appears to be minimized as $q \to 0$, shown in Figure 1(b) and 1(c). Whatever mechanism that lies behind these regularities is not investigated further and is beyond the scope of this paper. For the appropriate values of q, Table 1 then present some results from FLOM estimation of the parameters. The $\hat{\alpha}_{FLOM}$ estimator is seemingly unbiased with small standard deviations and a MSE that tends to zero throughout. This result holds for the symmetric distribution, with $\beta = 0$, as well as for the skewed distribution, with $\beta = 0.5$ and $\beta = 0.9$. While the estimator $\hat{\lambda}_{FLOM}$ performs well over most of the parameter space, $\hat{\beta}_{FLOM}$ becomes biased for $\alpha \geq 1.9$. The divergence can be understood from the fact that the characterization of GS distributions in (2) becomes symmetric

Table 1. Simulation results from FLOM estimation

(α, λ, β)		Mean $\beta = 0$	MSE	Mean $\beta = 0.5$	MSE	Mean $\beta = 0.9$	MSE
(0.5,1,β)	$\hat{\alpha}_{FLOM}$	0.500 (0.006)	0.000	0.501 (0.006)	0.000	0.500 (0.005)	0.000
	$\hat{\lambda}_{FLOM}$	1.002 (0.038)	0.001	0.999 (0.036)	0.001	1.001 (0.037)	0.000
	$\hat{\beta}_{FLOM}$	-0.000 (0.008)	0.000	0.500 (0.008)	0.000	0.900 (0.005)	0.000
(0.9,1,β)	$\hat{\alpha}_{FLOM}$	0.900 (0.010)	0.000	0.900 (0.007)	0.000	0.900 (0.006)	0.000
	$\hat{\lambda}_{FLOM}$	1.000 (0.022)	0.001	1.000 (0.030)	0.001	1.001 (0.045)	0.002
	$\hat{\beta}_{FLOM}$	-0.000 (0.002)	0.000	0.499 (0.025)	0.001	0.899 (0.014)	0.000
(01.1,1,β)	$\hat{\alpha}_{FLOM}$	1.100 (0.013)	0.000	1.100 (0.010)	0.000	1.100 (0.009)	0.000
	$\hat{\lambda}_{FLOM}$	1.001 (0.019)	0.000	0.999 (0.035)	0.001	0.995 (0.053)	0.003
	$\hat{\beta}_{FLOM}$	0.000 (0.003)	0.000	0.505 (0.067)	0.005	0.910 (0.138)	0.019
(1.5,1,β)	$\hat{\alpha}_{FLOM}$	1.501 (0.024)	0.001	1.501 (0.023)	0.001	1.500 (0.022)	0.001
	$\hat{\lambda}_{FLOM}$	1.000 (0.015)	0.000	0.999 (0.017)	0.000	1.000 (0.019)	0.000
	$\hat{\beta}_{FLOM}$	0.000 (0.024)	0.001	0.503 (0.050)	0.003	0.906 (0.084)	0.007
(1.9,1,β)	$\hat{\alpha}_{FLOM}$	1.902 (0.040)	0.002	1.903 (0.039)	0.002	1.901 (0.041)	0.002
	$\hat{\lambda}_{FLOM}$	0.999 (0.013)	0.000	1.000 (0.013)	0.000	1.000 (0.013)	0.000
	$\hat{\beta}_{FLOM}$	0.050 (1.076)	1.161	0.552 (2.593)	6.726	1.222 (8.806)	77.64
(2.0,1,β)	$\hat{\alpha}_{FLOM}$	2.002 (0.046)	0.002	2.001 (0.046)	0.002	2.003 (0.044)	0.002
	$\hat{\lambda}_{FLOM}$	1.000 (0.013)	0.000	1.000 (0.013)	0.000	0.999 (0.013)	0.000
	$\hat{\beta}_{FLOM}$	-1.597 (40.20)	1618	-3.374 (97.75)	9570	-0.302 (8.175)	68.27

Note: Results for $\hat{\alpha}_{FLOM}$, $\hat{\lambda}_{FLOM}$ and $\hat{\beta}_{FLOM}$ are obtained from 1000 samples with $n = 10000$ observations. For $\hat{\alpha}_{FLOM}$ the fractional moments $\mu_{\pm q}$ and $\mu_{\langle \pm q \rangle}$ are estimated with $q = 0.2$, whereas for $\hat{\beta}_{FLOM}$ and $\hat{\lambda}_{FLOM}$, they are estimated with $q = 0.01$. Standard deviations are presented in parenthesis.

Table 2. Simulation results from LM estimation

(α,λ,β)		Mean $\beta=0$	MSE	Mean $\beta=0.5$	MSE	Mean $\beta=0.9$	MSE
(0.5,1,β)	$\hat{\lambda}_{LM}$	0.978 (0.047)	0.003	1.002 (0.054)	0.003	1.003 (0.050)	0.003
	$\hat{\beta}_{LM}$	0.156 (0.077)	0.030	0.498 (0.054)	0.003	0.901 (0.048)	0.002
(0.9,1,β)	$\hat{\lambda}_{LM}$	0.995 (0.029)	0.001	1.000 (0.031)	0.001	0.999 (0.112)	0.013
	$\hat{\beta}_{LM}$	0.013 (0.006)	0.000	0.499 (0.025)	0.001	0.898 (0.020)	0.000
(01.1,1,β)	$\hat{\lambda}_{LM}$	0.998 (0.019)	0.000	1.001 (0.033)	0.001	0.999 (0.104)	0.012
	$\hat{\beta}_{LM}$	0.013 (0.006)	0.000	0.503 (0.065)	0.004	0.909 (0.131)	0.017
(1.5,1,β)	$\hat{\lambda}_{LM}$	0.999 (0.015)	0.000	0.997 (0.016)	0.000	1.000 (0.019)	0.000
	$\hat{\beta}_{LM}$	0.087 (0.037)	0.009	0.499 (0.047)	0.002	0.905 (0.078)	0.006
(1.9,1,β)	$\hat{\lambda}_{LM}$	0.997 (0.013)	0.000	0.998 (0.013)	0.000	1.000 (0.014)	0.000
	$\hat{\beta}_{LM}$	1.055 (3.126)	10.88	1.565 (17.72)	315.1	1.065 (1.393)	1.968
(2.0,1,β)	$\hat{\lambda}_{LM}$	0.999 (0.013)	0.000	0.999 (0.013)	0.000	0.997 (0.013)	0.000
	$\hat{\beta}_{LM}$	6.591 (39.59)	1609	34.76 (520.6)	$10^5\cdot 2.7$	8.766 (60.69)	$10^5\cdot 3.7$

Note: Results for $\hat{\lambda}_{LM}$ and $\hat{\beta}_{LM}$ are obtained from 1000 samples with $n = 10000$ observations, the using estimates of $\hat{\alpha}_{FLOM}$ as input from Table 1. Standard deviations are presented in parenthesis.

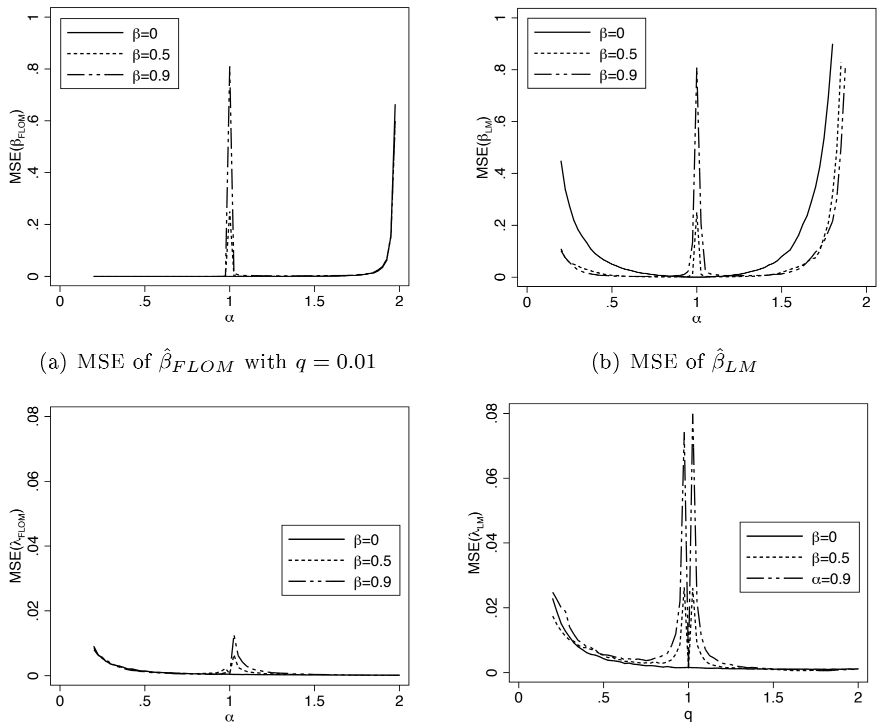
for $\alpha = 2$ as $\tan(\pi\alpha/2) = 0$.

The simulation results for the LM estimators are presented in Table 2, where $\hat{\alpha}_{FLOM}$ is used to estimate $\hat{\lambda}_{LM}$ and $\hat{\beta}_{LM}$. While MSE of $\hat{\lambda}_{LM}$ is close to zero, the skewness estimator $\hat{\beta}_{LM}$ blows up for values of $\alpha \geq 1.9$. The LM estimates of β, however, are slightly less accurate than the corresponding FLOM estimates when $\beta = 0$.

To better compare both methods with respect to $\hat{\beta}$ and $\hat{\lambda}$, Figure 2 shows their performance over the complete parameter space of α.[5]

For $\alpha = 1$ the fractional moments in (11) and (17) are not defined. As a consequence the β estimators diverges in that point, seen in Figure 2(a) and 2(b). Also worth noting is the behavior of $\hat{\beta}_{FLOM}$ and $\hat{\beta}_{LM}$ when α becomes large. Here, $\hat{\beta}_{FLOM}$ is found to explode in the single point when $\alpha = 2$. The $\hat{\beta}_{LM}$ estimator, on the other hand, is found to diverge much quicker. When it comes to the estimators of the scale coefficient λ, the FLOM and LM estimators behaves similarly. See Figure 2(c) and 2(d). They are both found to deviate somewhat for values of α in the vicinity of $\alpha = 1$. Although, the deviation is about four times smaller for $\hat{\lambda}_{FLOM}$ than it is for $\hat{\lambda}_{LM}$.

[5] Again, the FLOM estimate of α is relied upon in the LM estimators



(a) MSE of $\hat{\beta}_{FLOM}$ with $q = 0.01$

(b) MSE of $\hat{\beta}_{LM}$

(c) MSE of $\hat{\lambda}_{FLOM}$ with $q = 0.01$

(d) MSE of $\hat{\lambda}_{LM}$

Figure 2. Simulation results over α for appropriate q, 1000 simulations with $n = 10000$ and $\lambda = 1$

5 Concluding Remarks

This paper has considered the problem of parameter estimation of skewed geometric stable (GS) distributions. Despite a growing interest in heavy tailed distributions, finding suitable estimators have restricted their practical implementation (Kozubowski, 2001). To address this problem, easily implemented estimators were derived from the fractional moments of GS distributions. The resulting estimators extend the symmetric estimators previously developed by Cahoy (2012) and Kozubowski (2001) and are based on the methods of fractional lower order moments (FLOM) and logarithmic moments (LM).

Simulation results show that the FLOM estimators, which perform well over most of the parameter space, has superior performance to the LM estimators that are found to be more restricted.

To conclude, future studies could benefit from investigating further applications of the GS distribution to empirical data, which should be facilitated by the estimators developed in this paper. One potentially fruitful area is within the research of firm growth, where recent advances finds the growth rate distribution to be characterized by a Laplace type distribution with Pareto tails (Fu et al, 2005), much similar to a GS distribution. Furthermore, the present paper only considers the univariate distribution. As has been pointed out elsewhere, it would also be of interest to extend the estimators to the multivariate distribution that has many applications in e.g. finance (Cahoy, 2012).

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